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Determination of mean and dynamic skin friction, separation and transition in low-speed flow with a thin-film heated element

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The application of a heated thin metallic film to the measurement of mean skin friction in laminar and turbulent flow on a flat plate, circular cylinder and in an annular tunnel is described. The manner in which transition and separation are detected with this instrument is illustrated by reference to tests on a circular cylinder. The influence of the ambient air temperature and the gauge temperature on the behaviour of the instrument is analysed and it is shown that a reliable element can be constructed, capable of being moved from one location to another, while retaining its calibration. An element, calibrated on an oscillating flat plate, is used to obtain the spectral density of skin friction in a turbulent boundary layer.

1. Introduction

The usual methods of skin-friction measurement are by means of surface tubes, floating elements and heated elements. Surface tubes have a narrow opening, which must be aligned with the flow direction, and which must project into the boundary layer. This necessitates small size and accurate machining and positioning to give readings close to the surface and cause the minimum flow disturbance. For example, a very small error in positioning may account for a difference of the order of 30% in the results for the skin-friction coefficient at $\mathrm{Re} = 10{\cdot}6 \times 10^4$† for the 3 in. and 6 in. diameter cylinders discussed by Fage & Falkner (1931*a*). A razor blade placed over an ordinary pressure hole simplifies the technique, and although the assembly cannot be removed for calibration, it may be applicable to turbulent flow, provided that a universal relation for the boundary-layer profile holds good. If separation is to be detected, calibration of the surface tubes is usually unnecessary as all that is required is to note when its reading equals the local static pressure. The slow response of the instrument makes it unsuitable for instantaneous measurements in unsteady flow.

Floating element techniques appear to have been applied only to pipes and flat plates. The element becomes part of the surface, and friction on the element is balanced mechanically or electromagnetically. The equipment, which must be housed inside the model, is cumbersome, and the technique cannot measure skin friction at a point, but averages it over the area of the element, which must be

† Reynolds number based on cylinder diameter.

large enough for the effects of the necessary gap around it not to dominate the results. Because of inertia effects, dynamic measurements are again impossible.

Oil film techniques give a qualitative picture of separation and transition, but wind tunnels must frequently be stopped when these techniques are being applied, and their application to unsteady problems is impossible.

The probe, whose development is described in the present paper, consists of a thin (approximately 1μ) platinum–silver–alloy film baked onto a Pyrex-glass substrate. These films have a temperature coefficient of about $1 \times 10^{-3}/°C$, and their use as surface resistance thermometers is obvious. They have found wide application as heat-transfer gauges in shock tunnels; for this, the film is electrically heated such that the heat transfer from the flow to the probe greatly exceeds the electrical heating of the probe. The probe voltage is fed into an electrical analogue of the substrate conduction equation, which immediately gives the dynamic-heat-transfer rate.

When used to measure skin friction, the probe must act as both a heat source and a heat-transfer gauge. The heat supplied to the air is related to the local value of skin friction, and can be deduced from the electrical power supplied to the probe. A thin film of thickness 1μ has a response time of about $0{\cdot}04\,\mu$sec (Schultz 1963), so it is eminently suited for dynamic measurements. Since the thermal- and velocity-boundary-layer equations are decoupled for the small amounts of heat involved, the heated thin film causes no disturbance to the boundary layer and, by using a film of circular planform, the instrument is insensitive to flow direction. Films can be used in air or water and the considerable experience that has been gained in recent years in using them in shock tunnels as heat-transfer gauges (Schultz 1963) has shown that they are robust and simple to make.

Work on various forms of heated element has been carried out by several authors (Fage & Falkner 1931*a*, *b*; Ludwieg 1950; Liepmann & Skinner 1954; Myers, Schauer & Eustis 1963). Fage & Falkner employed heated-foil elements, and Myers *et al.* a thin-platinum-film strip $\frac{1}{4}$ in. long, mounted in a long $\frac{3}{4}$ in. diameter insulating disk. Without further development, none of these probes can be used at successive positions on any model other than one of constant curvature; and, if one probe were to be made for each curvature, the difficult problem of calibration must be solved. Also, it is assumed that the response times were relatively long (except for the probe described by Myers *et al.*) although none of the authors attempted dynamic measurements.

Liepmann & Skinner (1954) studied both laminar and turbulent flow over a flat plate. It was assumed that the effective width L of the film must greatly exceed the thermal-boundary-layer thickness for boundary-layer treatment to be permissible, and that the thermal boundary layer is thinner than the laminar sublayer of turbulent flow.

For a probe to be applicable both to laminar and turbulent flow (with a single calibration), this leads to the criterion

$$1 \ll \frac{Q_w L}{k \Delta T_0} < \frac{\sigma}{C_f},$$

where Q is the heat flux, k the thermal conductivity, ΔT_0 the temperature difference of film and stream, σ the Prandtl number and C_f the skin-friction coefficient.

Liepmann & Skinner have shown, by dimensional analysis, that

$$i^2R/\Delta T_0 \propto \tau_w^{\frac{1}{3}},$$

where i is the current, R the resistance and τ_w the wall shear stress. Their experimental results confirm this, although the worst data indicate a possible 60% error in skin-friction coefficient if it were determined in this manner. However, much of the error might well have arisen from the boundary-layer measurements (performed with hot wires), which were used to give the standard value of skin friction.

In view of the scatter of the results of Liepmann & Skinner, and in view of Bradshaw & Gregory's (1961) work on the difference in calibrations in laminar boundary layers and the laminar 'sublayer' of turbulent boundary layers, the unique calibration of Liepmann & Skinner must be treated with caution.

2. Construction of the gauges

Two types of gauge were used, the first of which was baked onto the surface of a 2 in. Pyrex-glass cylinder. The films were 0·375 in. long and 0·05 in. wide; the cylinder comprised three lengths, the centre of which was 1 in. long, and carried the films. Conducting paint led from the film ends to the inside of the cylinder,

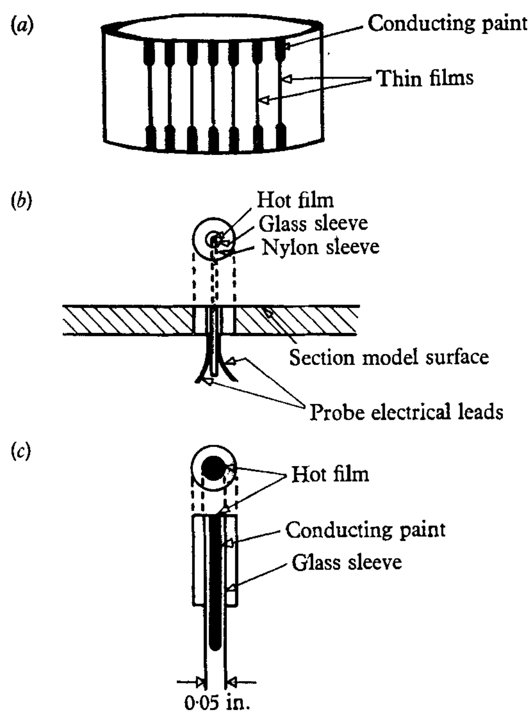

FIGURE 1. Heated-element gauges. (*a*) Pyrex-glass ring; (*b*) removable probe; (*c*) enlarged view of the removable probe.

where soldered connexions were made to the film leads (see figure 1 (*a*)). Attempts to match gauges failed, because two parameters have to be matched ($Q/\Delta T_0$ and L) with one adjustment, that of operating resistance. Gauges that cannot easily be calibrated, like those above, will be of no use for measurements on any body other than a circular cylinder or flat plate, so a small dismountable element has been developed to overcome this restriction.

This proved to be very simple to make, despite its small size, and has operational advantages over the larger gauges. It can be removed and replaced and, without great attention to displacement relative to the surface, it has been found that the readings are repeatable to a high order of accuracy.

Pyrex-glass rod of 0·050 in. diameter was cut into 0·75 in. lengths, the ends ground and then smoothed by heating in a gas flame. A thin platinum film was painted on as shown in figures 1 (*b*), (*c*) and was fired at 620 °C for 30 min in a ventilated electric furnace. To facilitate soldering, and minimize heat losses to the probe mounting, Johnson–Matthey double coat conducting paint was applied to the film extremities. The probe was encased in a ground-Pyrex-glass cylindrical sleeve to minimize heat losses, and the assembly mounted on a $\frac{3}{8}$ in. diameter plug.

3. Skin-friction measurement in a laminar boundary layer

The thermal-energy integral equation for steady laminar flow is solved in appendix 1 by Curle's (1962) method.

The final equation of appendix 1 is

$$ak^2\tau_w\left(\frac{\Delta T_0}{Q_w}\right)^3-\tfrac{1}{2}b\frac{dp}{dx}k^3\left(\frac{\Delta T_0}{Q_w}\right)^4=-\frac{\mu^2L}{\rho k\sigma}, \qquad (1)$$

where a and b are constants dependent on the boundary-layer profile, p, μ and ρ are the pressure, viscosity and density respectively. If the pressure gradient is small enough to regard the temperature profile as linear, we obtain

$$\tau_w^{\frac{1}{3}}=\left(\frac{\mu^2}{\rho\sigma aL^2}\right)^{\frac{1}{3}}\left(\frac{-Q_wL}{k\Delta T_0}\right). \qquad (2)$$

For a given film, $-Q_wL/(k\Delta T_0)=A'(i^2R/\Delta T_0)+B'$, where A', B', $\mu^2/(\rho\sigma aL^2)$ are functions of ΔT_0, and B' is associated with heat losses to the film mounting. For small variations in ΔT_0

$$\tau_w^{\frac{1}{3}}=A(i^2R/\Delta T_0)+B, \qquad (3)$$

where A, B, are constant.

4. The circular-cylinder measurements

Steady skin friction

The purpose of the work described in this section was to check the validity of equation (3). One film only was used, and the circular cylinder was rotated to give readings at different values of θ, the angle between the stagnation point and the gauge position. The film was controlled by a constant-temperature feedback bridge,† and the film current was measured (by means of bridge voltage) by a potentiometer of four-figure accuracy. Readings of film current were taken at Reynolds numbers of 6×10^4, $8\cdot3 \times 10^4$ and $10\cdot6 \times 10^4$ for points in the range $0 \leqslant \theta \leqslant 180°$.

The Reynolds numbers were chosen so that the heat-transfer readings could be compared with the experimental results of Fage & Falkner (1931*b*) but for reasons given in the introduction, the 3 in. cylinder results have been rejected in favour of a theoretical solution (using the technique of Falkner & Skan 1930), based on their static pressure curves reported by Fage & Falkner (1931*a*). The results are displayed on the graph of $i^2R/\Delta T_0$ versus $\tau_w^{\frac{1}{3}}$ (figure 2); because of the linearity of this graph, two points will be sufficient to fix the calibration line.

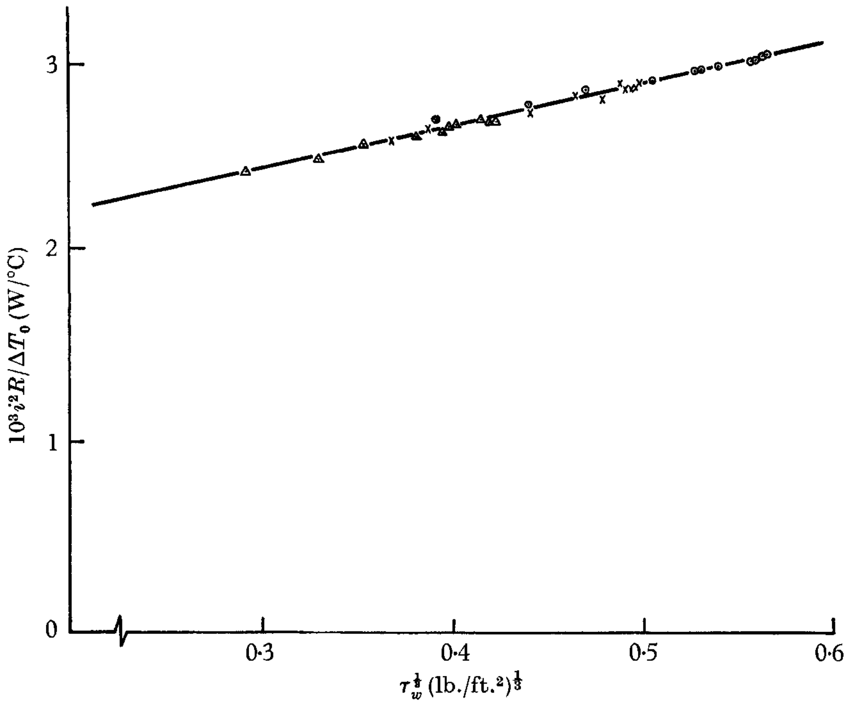

FIGURE 2. Mean skin friction on a circular cylinder measured with a thin film in a laminar boundary layer. $10° \leqslant \theta \leqslant 60°$. $\odot$, Re $= 10\cdot6 \times 10^4$; $\times$, Re $= 8\cdot3 \times 10^4$; $\triangle$, Re $= 6 \times 10^4$.

Laminar separation

On a cylinder, similar to that which carried the thin films, a static-pressure tube and a Stanton tube were mounted. When the pressure levels in the two tubes were identical, it was assumed that separation had occurred. These points are marked S on the curves (figures 3, 4) and are seen to lie close to the minima of the i versus θ curves. The film voltage fluctuations, shown in figure 5, also indicate separation near $\theta = 80°$. For these measurements, it was necessary to heat the thin film with batteries to achieve an acceptable noise level; the film voltage was displayed on an oscilloscope and recorded photographically.

† DISA constant-temperature anemometer 55 AO 1.

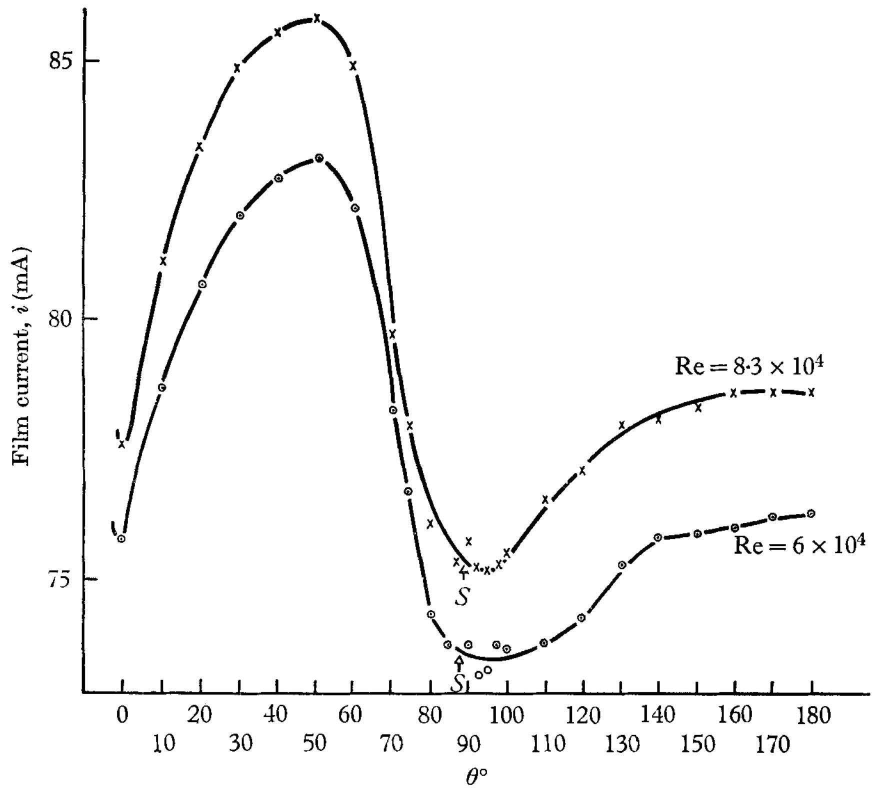

FIGURE 3. Mean current values for a thin film on a circular cylinder for a laminar boundary layer.

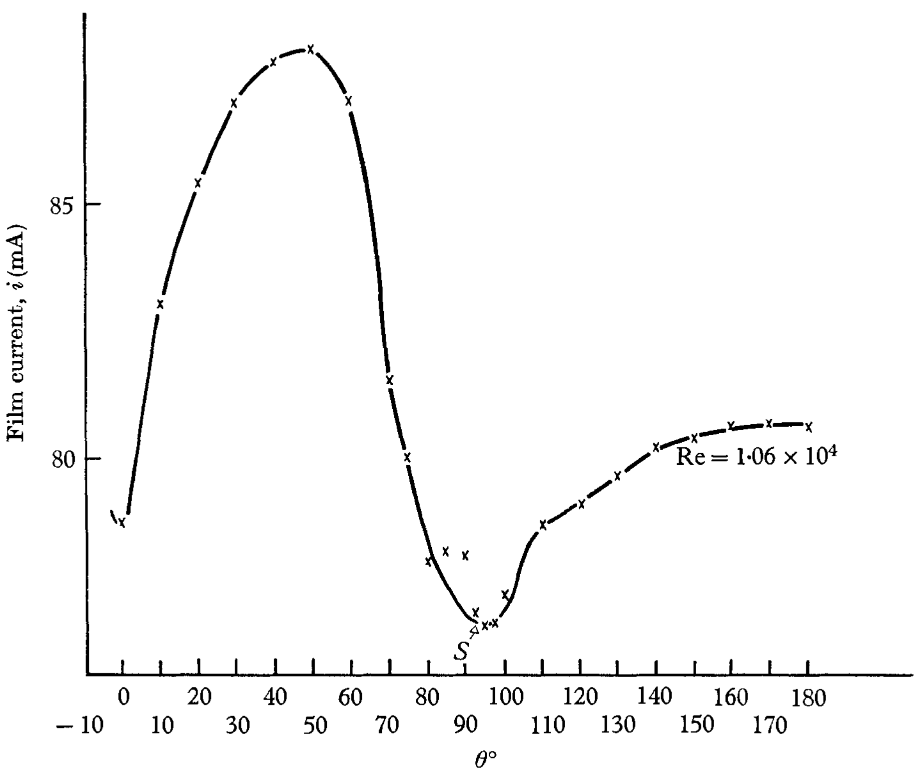

FIGURE 4. Mean current values for a thin film on a circular cylinder for a laminar boundary layer.

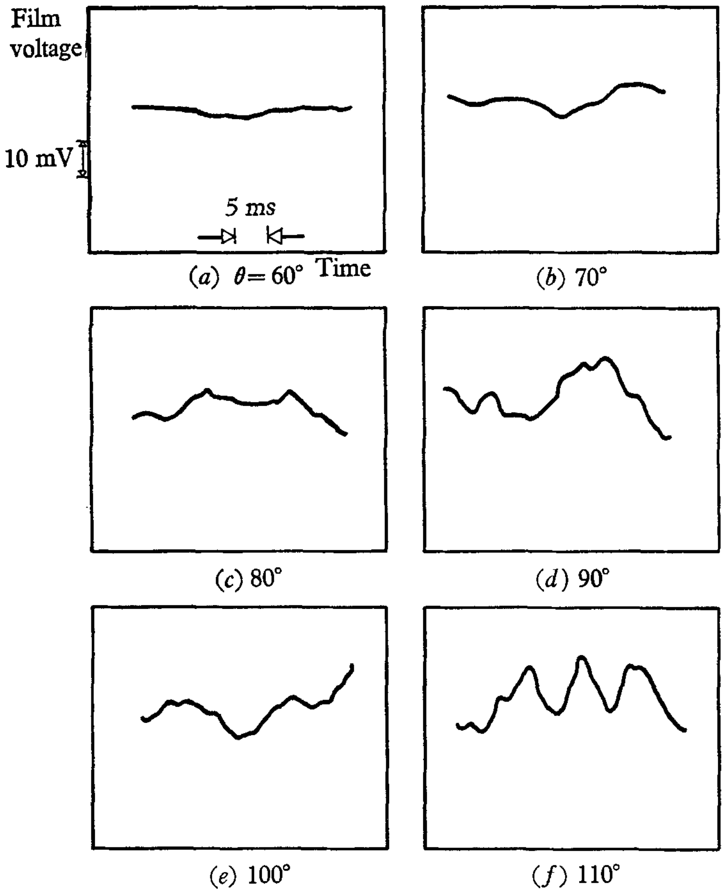

FIGURE 5. Fluctuating component of the film voltage for a laminar boundary layer. $\mathrm{Re} = 8{\cdot}3 \times 10^4$.

Turbulent boundary layer

The mean level of heat transfer is related to the random fluctuations of the velocity field, and so the laminar-separation criterion (of minimum mean heat transfer) cannot be expected to hold in turbulent flow. However, the mean current curves in turbulent flow (figure 6) are still of interest. For $\mathrm{Re} = 10{\cdot}6 \times 10^4$ and $14{\cdot}4 \times 10^4$ the sharp decreases in film current start at $\theta = 100°$ and $105°$, respectively, which correspond to surface-tube indications of $108°$ and $113°$ and a surface-oil-film line at about $110°$. The minima near $\theta = 160°$ are matched by a second oil-film line; this, however, is well within the separated region.

The records of figure 7 indicate a large and sudden increase in voltage fluctuations at $\theta = 120°$ and this may be a better separation criterion.

Transition

With the surface of the cylinder slightly roughened, the i versus θ curves in figure 8 are similar to those for laminar flow until the laminar separation point is

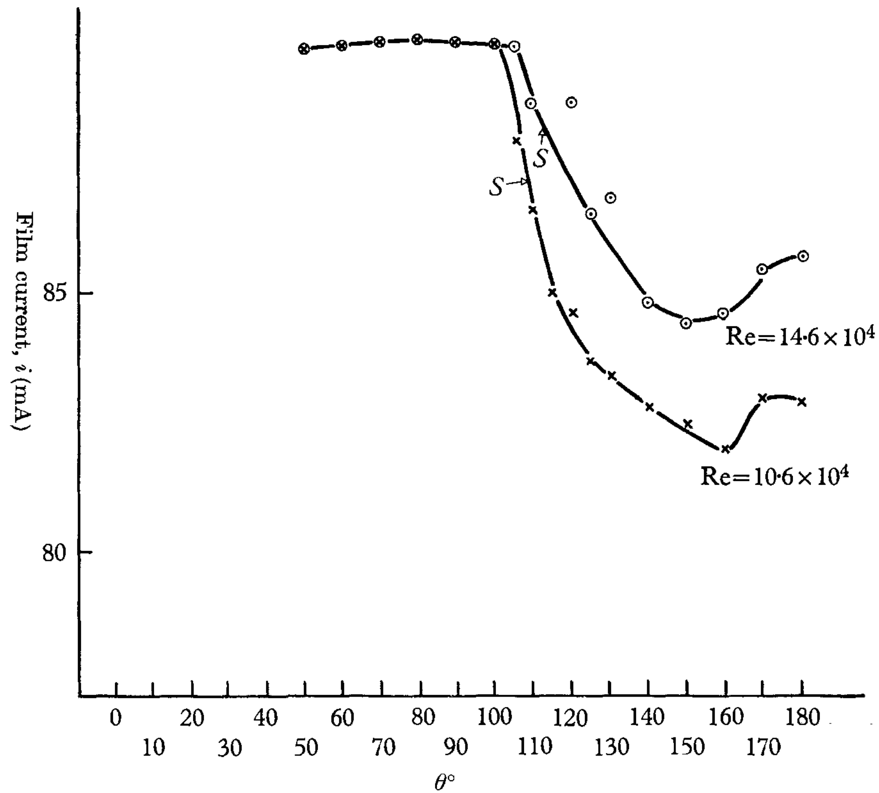

FIGURE 6. Mean current values for a thin film on a circular cylinder for a turbulent boundary layer.

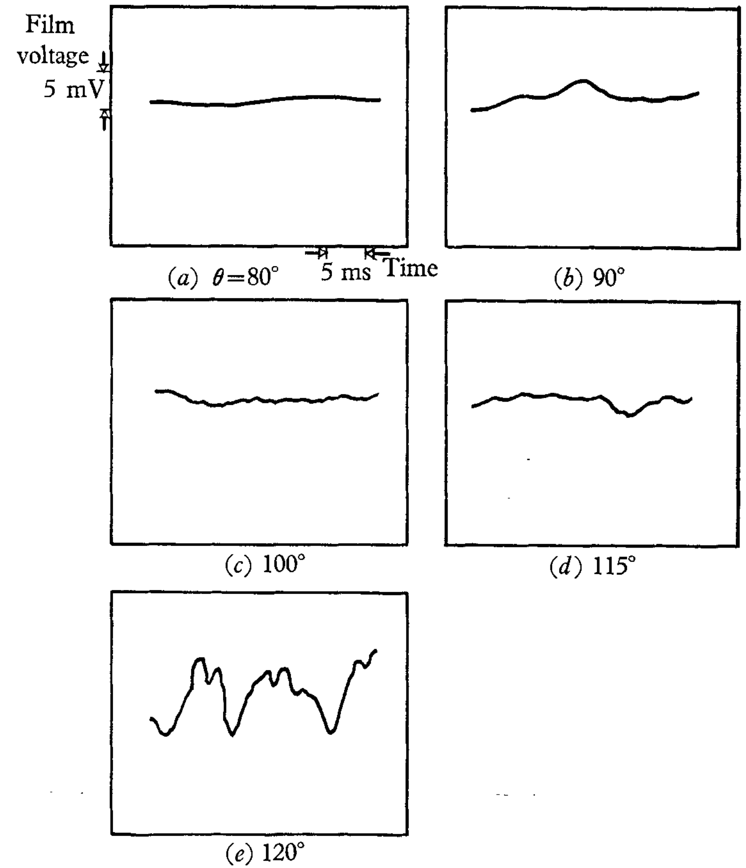

FIGURE 7. Fluctuating component of the film voltage for turbulent boundary layer. Re = $8{\cdot}3 \times 10^4$.

approached. After the minimum has occurred (rather sooner than in the laminar case), the curve rises sharply to a new maximum. An explanation is that the boundary layer has undergone transition, and reattached to the surface before finally separating as a turbulent layer. The records of figure 9 indicate large film voltage oscillations for $\theta \geqslant 80°$ which stabilize at 100° and become large again when $\theta = 140°$. The stability at 100° matches the second peak of the i versus θ curve of figure 8.

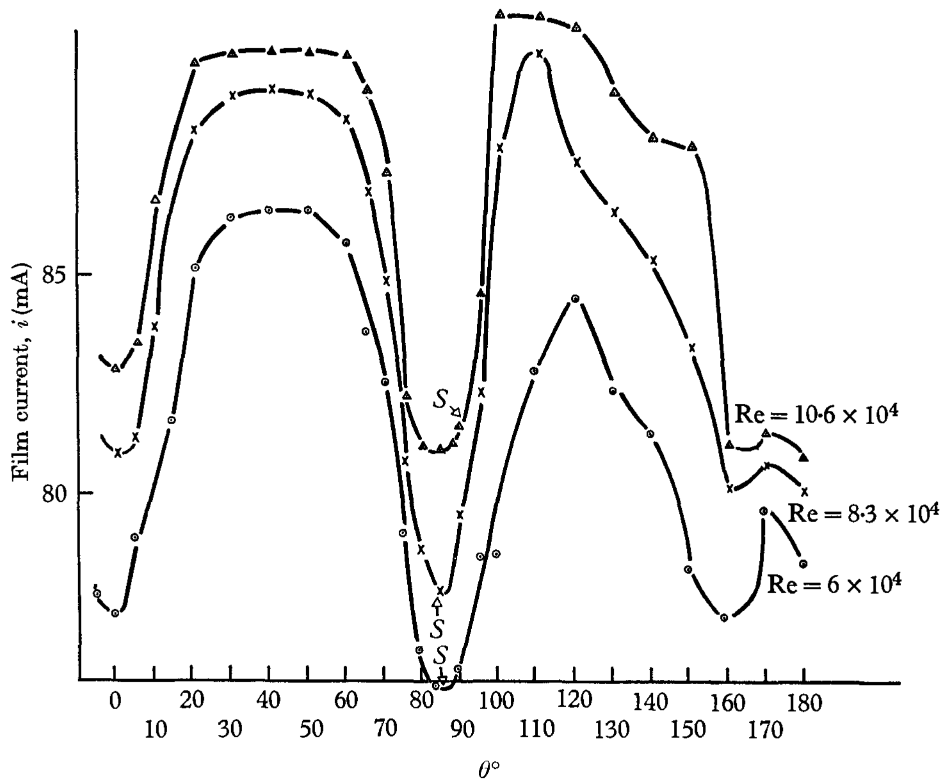

FIGURE 8. Mean current values for a thin film on a circular cylinder for flow with transition.

Dynamic measurement of skin friction

Since the diffusion time of a thin platinum film of thickness 1 μ is of the order of 0·04 μsec, the gauges are highly suitable for dynamic measurements, if suitable recording equipment is used.

Traces obtained with an ultra-violet recorder (figure 10) indicate the dynamic readings obtained on a circular cylinder. When no splitter plate was present the periodic component was observed, and frequency analysis showed that its frequency was approximately constant with only small random changes. The periodic fluctuations are attributed to the presence of a vortex street, and when this was suppressed by means of a splitter plate spectrum analysis of the readings showed no periodicity.

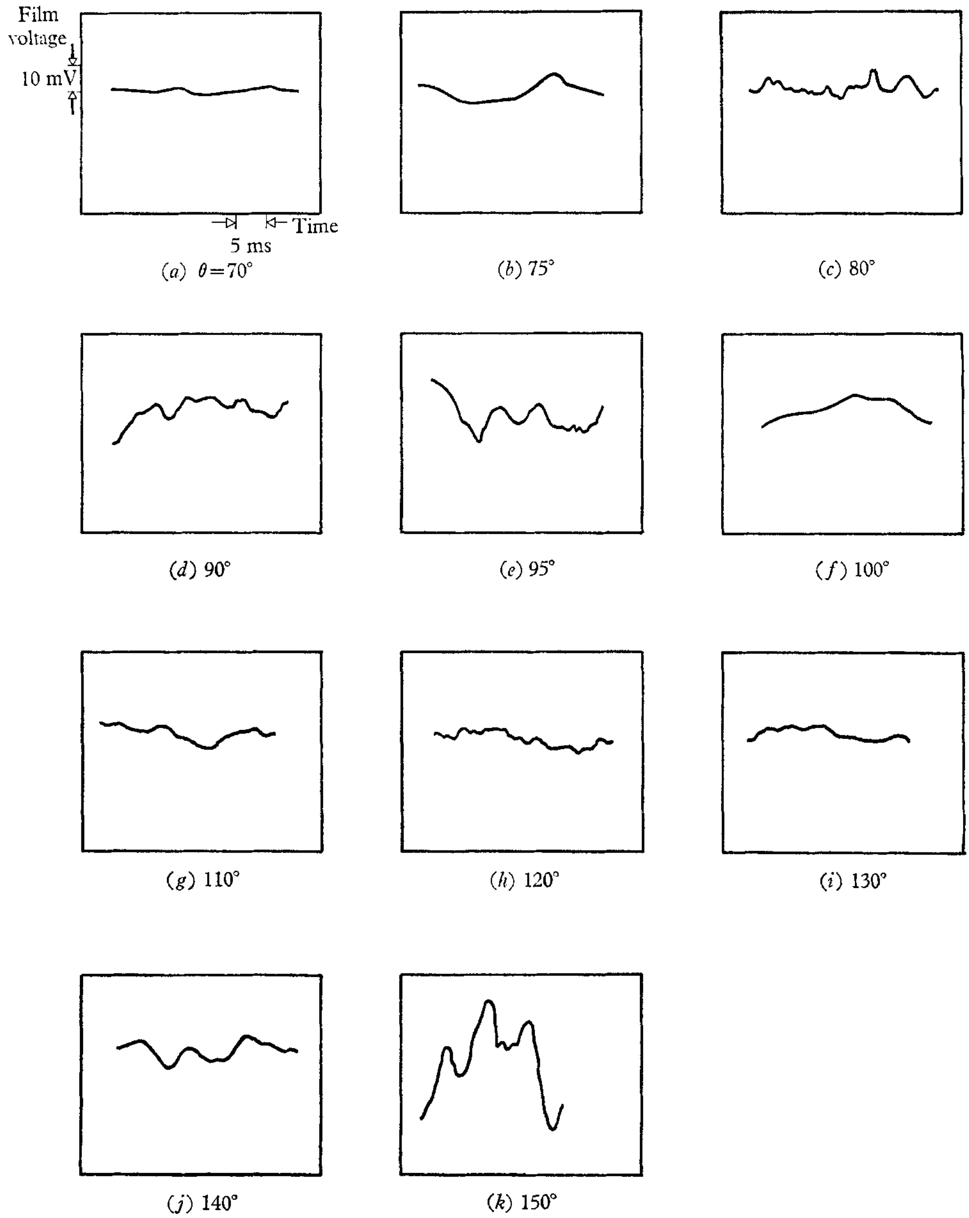

FIGURE 9. Fluctuating component of the film voltage for flow with transition. $Re = 8{\cdot}3 \times 10^4$.

5. Turbulent annular-tunnel measurements

Accurate theoretical solutions of the turbulent boundary layer on a circular cylinder do not exist, but an annular tunnel overcomes these difficulties, provided the flow is uniform. (Pitot-static traverses have indicated uniformity of flow in the tunnel used.) It can be shown that $\tau_w^{\frac{1}{3}} = 6{\cdot}284 \times 10^{-2}(\Delta p)^{\frac{1}{3}}$, where Δp is the pressure drop in p.s.i. over a 72·5 in. length of tunnel.

A glass-sleeved probe was set flush with the tunnel wall, and heat-transfer coefficients for various values of Δp were obtained, and the results shown in figure 11 are typical. Provided the thermal-layer thickness is less than the laminar sublayer of turbulent flow, equation (3) can be expected to hold; and this is supported by all the results taken, which included perhaps a hundred separate runs.

In the early stages of the investigation, it was not appreciated how significant the changes in ambient temperature could be. The thin film is, of course, primarily a thermometer, and ambient temperature changes of 0·1 °C are significant. For changes of up to 4 °C, the coefficients A, B may be treated as constant, provided ΔT_0 is corrected in

$$\tau_w^{\frac{1}{3}} = A(i^2R/\Delta T_0)+B.$$

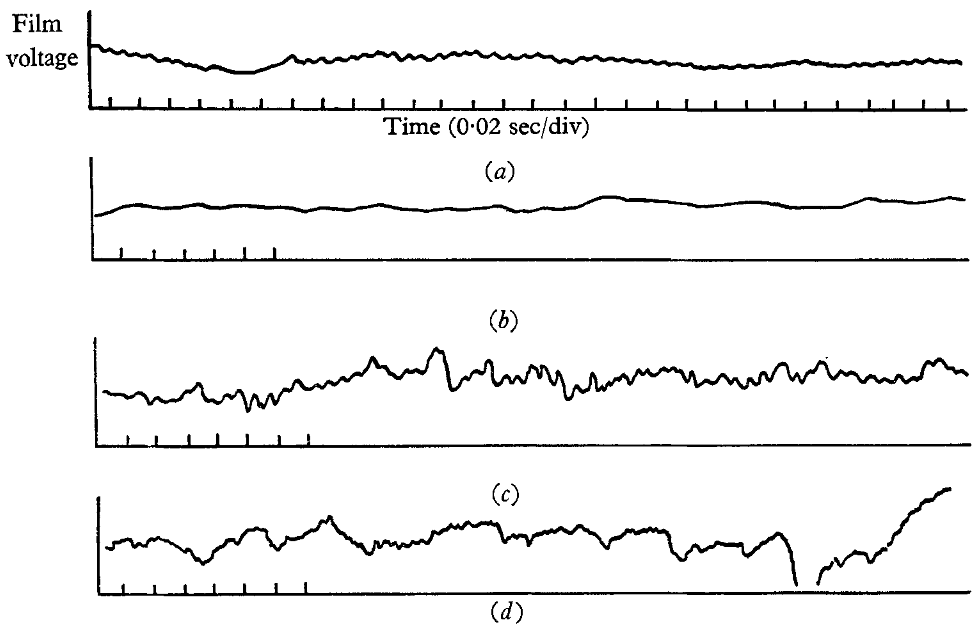

FIGURE 10. Fluctuating component of the film voltage for a thin film on a circular cylinder, with and without a splitter plate, for laminar flow. (*a*) $\theta = 40°$ without splitter; (*b*) $\theta = 40°$ with splitter; (*c*) $\theta = 80°$ without splitter; (*d*) $\theta = 80°$ with splitter.

Correction for large changes of temperature differences

In flows where heat transfer exists, or temperature changes are large, A and B must be determined as functions of ΔT_0. This calibration would be necessary for work in high-speed flow.

The flow velocity, and hence the skin friction, was kept constant while the film was operated at different temperature differences. The curves of iV against ΔT_0 in figure 12 are clearly non-linear, although they deviate only slightly from the linear. A quadratic, fitted by the least squares method, would appear to be adequate. These measurements support the analysis of appendix 2. From these curves the calibration curve for all temperatures of the probe, that is, for any value of $Z = \{(R/R_0) - 1\}\, 10^3 = 10^3 \alpha \Delta T_0$ is

$$10^3 Vi = 10^2 \tau_w^{\frac{1}{3}} (0{\cdot}11944Z - 0{\cdot}7686 \times 10^{-3} Z^2) + (0{\cdot}7661Z + 0{\cdot}0087Z^2).$$

For a probe of different construction, however, the constants will be different.

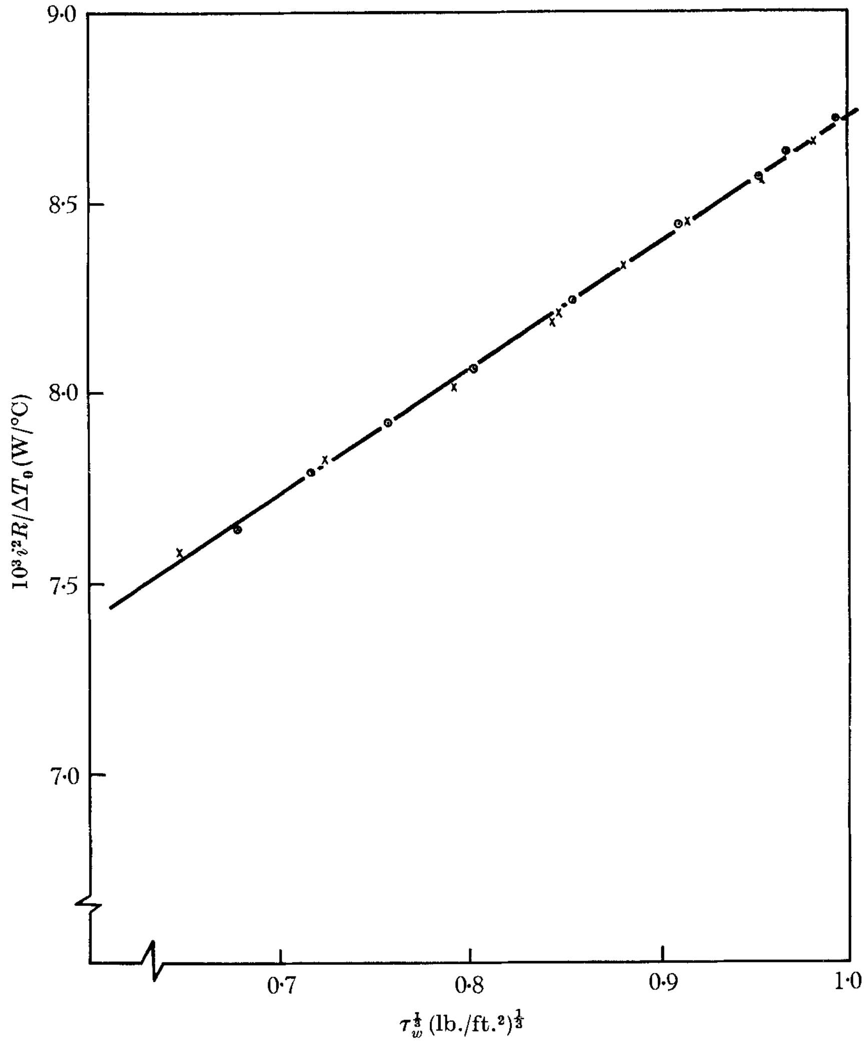

FIGURE 11. Mean skin friction in a turbulent-flow annular tunnel, measured with flush-mounted thin film. ×, First set of readings; ⊙, second set of readings after probe removed and replaced.

Probe lying on the surface

Although the flush-mounted probe causes least flow disturbance, it may occasionally be more convenient to use a probe lying on the surface, in a manner similar to a Preston tube. Two graphs of $i^2R/\Delta T_0$ against $\tau_w^{\frac{1}{3}}$ obtained on the

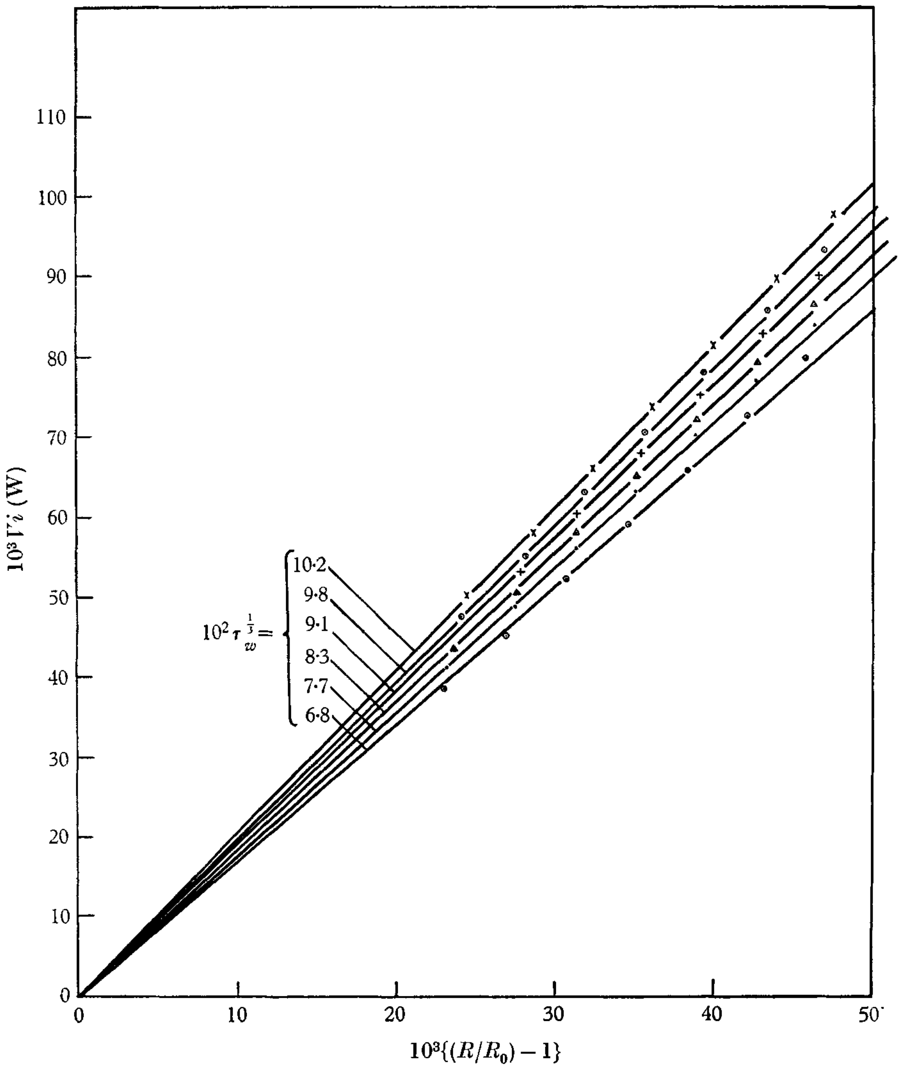

FIGURE 12. Mean skin friction in a turbulent-flow annular tunnel, measured by a thin film over a range of operating temperatures.

inside and outside of the annulus are distinct, and are straight lines (figure 13). The method may be of value in thick turbulent boundary layers.

It is interesting to note that $i^2R/\Delta T_0$ is still proportional to $\tau_w^{\frac{1}{3}}$ and not some other power of τ_w determined by the outer flow. This can be explained if the deformed flow produces a laminar sublayer over the film surface, and if the skin friction so produced is linearly related to the skin friction in the absence of the probe.

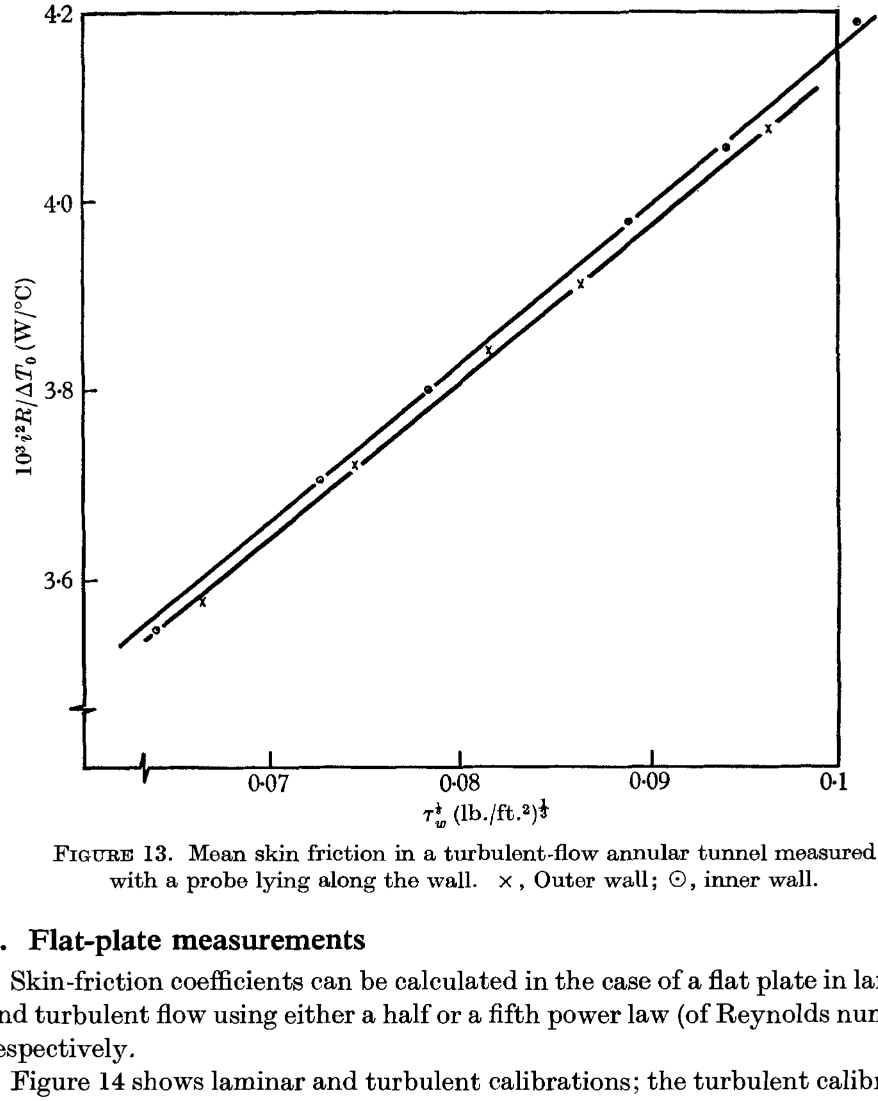

FIGURE 13. Mean skin friction in a turbulent-flow annular tunnel measured with a probe lying along the wall. ×, Outer wall; ⊙, inner wall.

6. Flat-plate measurements

Skin-friction coefficients can be calculated in the case of a flat plate in laminar and turbulent flow using either a half or a fifth power law (of Reynolds number), respectively.

Figure 14 shows laminar and turbulent calibrations; the turbulent calibration is made up of points from two positions on the plate, showing the resetting error to be negligible. In turbulent flow, no errors in $i^2R/\Delta T_0$ were introduced by raising or lowering the probe 3 thousandths of an inch in a direction perpendicular to the plate. This could indicate that the plug may be used at points in models with a radius of curvature not less than 6 in. For laminar flow, repositioning the probe was equally easy (figure 15), but the raising and lowering of the probe produced significant effects, although these could be greatly reduced by fairing-in the plug. The laminar and turbulent calibration lines are distinct. The error incurred in τ_w if one calibration were used for both laminar and turbulent boundary layers would be about 14 %, while the scatter is usually less than 5 % of τ_w. This agrees with the conclusions of Bradshaw & Gregory (1961) who used a Stanton tube and a hot wire near the surface to compare laminar and turbulent calibrations.

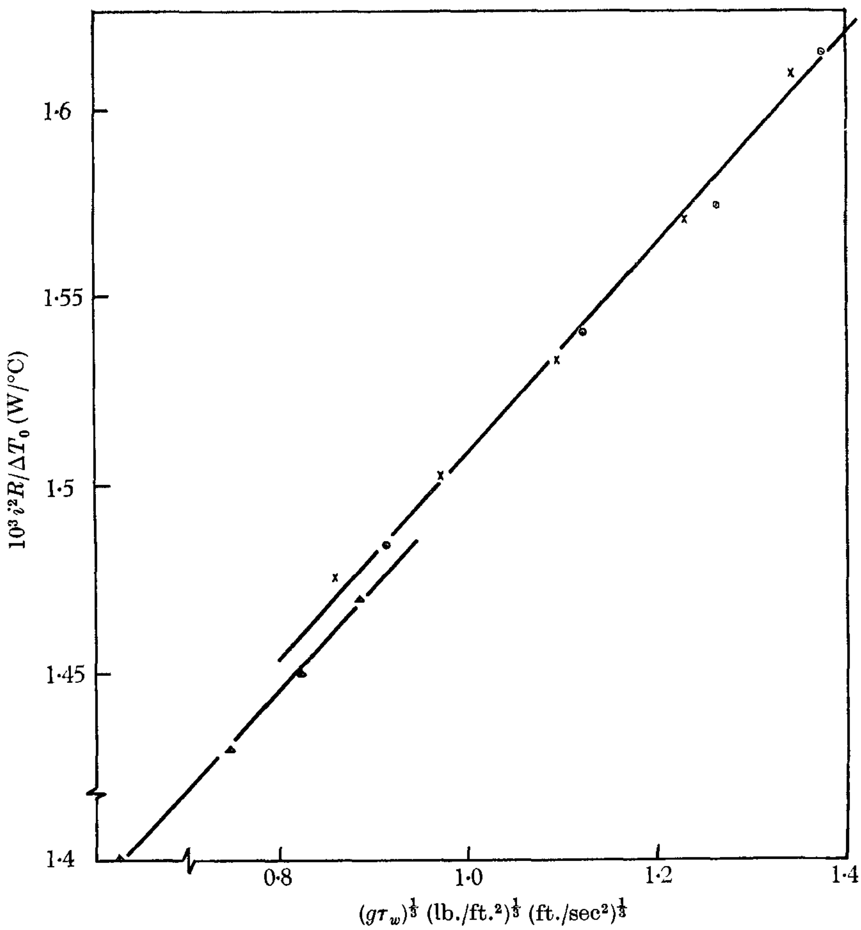

FIGURE 14. Mean skin friction measured by a thin film in laminar and turbulent boundary layers on a flat plate. Distance from the leading edge: ×, 6·6 in. (turbulent); ⊙, 4·6 in. (turbulent); △, 4·6 in. (laminar).

The calibration of probes in position

It would be preferable to calibrate the instrument in position, so that the same value of skin friction is obtained over both the heated thin film and the calibrating instrument. It is difficult, however, to vary the position of all but a Preston tube, and thus Liepmann & Skinner (1954) employed boundary-layer measurements performed with a hot-wire anemometer to deduce skin friction. The analysis of appendix 3 indicates that a hot wire calibrated in uniform steady flow may not be accurate when the steady flow is non-uniform, as it is in a boundary layer. If measurements are made near the wall, then the error is greatest. It is interesting to note that, if the results of Bradshaw & Gregory (1961) are re-plotted as Nu against $\tau_w^{\frac{1}{3}}$ instead of Nu against $\sqrt{\tau_w}$ ($\propto \sqrt{u}$), the curves become straight lines. The results were obtained with a hot wire about 0·002 in. from the surface, and it would appear that the skin-friction term dominates the velocity term even at that distance from the surface.

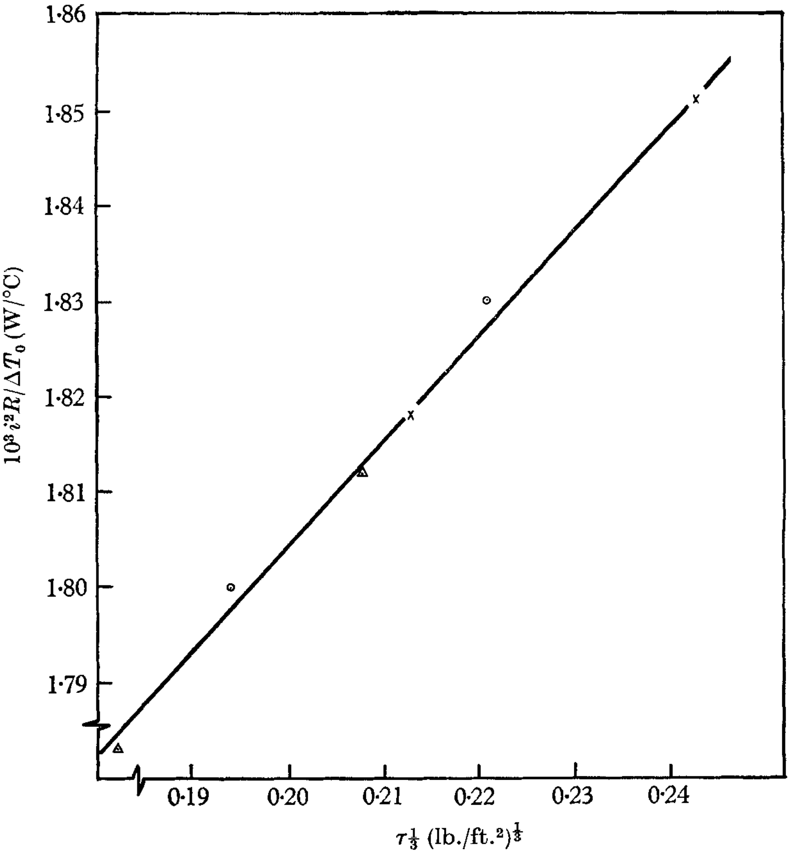

FIGURE 15. Mean skin friction in laminar flow over a flat plate. The effect of transplanting the thin-film gauge. Distance from the leading edge: ×, 2·6 in.; ⊙, 4·6 in.; △, 6·6 in.

Unsteady skin friction due to a turbulent boundary layer

In a laminar incompressible boundary layer, the thermal energy can be taken to be

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=(k/C_p\rho)\frac{\partial^2 T}{\partial y^2}.$$

In unsteady flow, $\partial T/\partial t$ may be as important as $u(\partial T/\partial x)$, and this will be so if $\omega L/U_0$ is of order unity, where ω is the frequency in radians/sec, and U_0 the main stream velocity.

For sufficiently small values of $\omega L/U_0$, it might be expected that the steady-state calibration of $Q_w \propto \tau_w^{\frac{1}{3}}$ will hold instantaneously.

Thus $$dQ_w/Q_w = \tfrac{1}{3}d\tau_w/\tau_w,$$

and writing $$Q_w = Q_0+Q_1\exp(i\omega t),$$
$$\tau_w = \tau_0+\tau_1\exp(i\omega t),$$

this may be expressed $$Q_1/Q_0 = \tfrac{1}{3}\tau_1/\tau_0. \qquad (4)$$

If the frequency parameter $\omega L/U_0$ is significant, then Q_1 and τ_1 may no longer be in phase and (4) will need to incorporate a function of $\omega L/U_0$. The form

$$|Q_1/Q_0| = |\tau_1/\tau_0| f(\omega L/U_0) \tag{5}$$

can be supported by a perturbation theory based on Curle's method for steady boundary layers, provided the Prandtl number and probe position remain constant.

The probe was fixed in a flat plate which could be oscillated in its own plane in the stream direction. The film was operated at constant temperature, and the voltage signal (of the feedback bridge) was fed to a wave analyser. The fluctuating skin friction was calculated with Lighthill's (1954) formulae, and the probe was calibrated over a range of frequencies below 100 c/s, and over a range of mean flow velocities between 30 and 100 ft./sec.

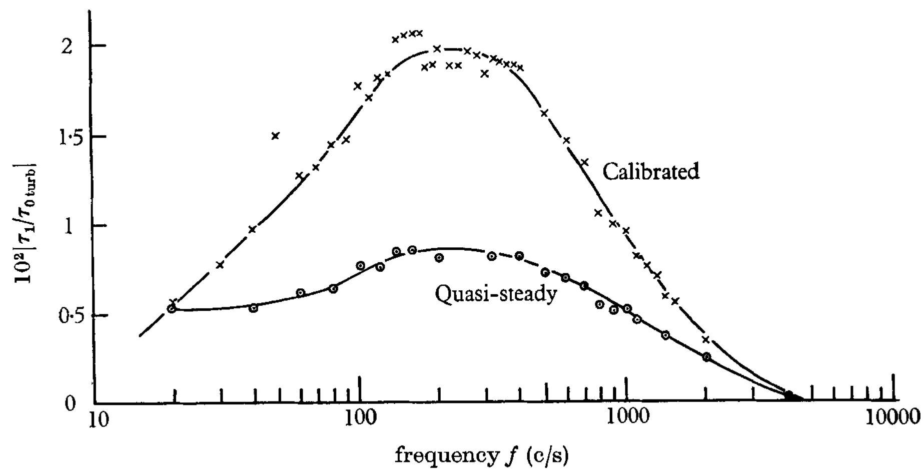

FIGURE 16. Fluctuating skin-friction measurement in a turbulent boundary layer.

It was found that $f(\omega L/U_0)$ could be taken to be linear, at least within the range of frequencies over which the probe was calibrated.

A trip wire was attached to the plate near its leading edge, and the probe was statically calibrated in the resulting turbulent boundary layer, and then the dynamic signal was harmonically analysed for a fixed flow velocity. The graph of $10^2|\tau_1/\tau_0|$ against frequency in figure 16 was computed from the spectral density of the dynamic signal in the turbulent boundary layer, and the dynamic calibration obtained below 100 c/s on an oscillating plate with a laminar boundary layer. A quasi-steady calibration, obtained from the static calibration in the turbulent boundary layer, is included for comparison.

A hot-wire probe was placed immediately above the hot-surface element, and its signal was recorded and the fluctuating velocity u_1 was calculated, assuming a quasi-steady law. (In the case of a fine hot wire, the parameter $\omega L/U_0$ is negligible up to high frequencies if U_0 is sufficiently large.) A graph of $\frac{1}{2}(u_1/U_0) \times 10^3$ against frequency is shown in figure 17. The shape of the curve is similar to those of figure 16 although the maximum occurs at a higher frequency.

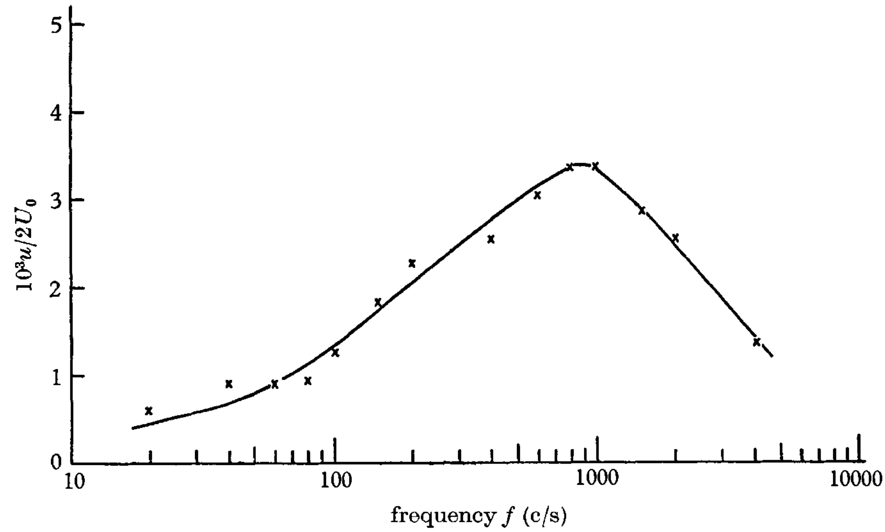

FIGURE 17. Fluctuating-velocity measurement in a turbulent boundary layer.

It is possible that the calibration obtained for up to 100 c/s is inadequate above this frequency, but this can easily be extended once a higher-frequency calibration rig has been built.

7. Conclusion

It has been demonstrated that a flush-mounted hot-film probe can be used for skin-friction measurements in both turbulent and laminar boundary layers, with errors of less than about 5 %, and with two-point calibration. A single flat-ended probe can be transplanted to positions with radius of curvature from 6 in. to infinity without loss of accuracy, at least in turbulent boundary layers. The instrument is capable of detecting small high-frequency fluctuations in skin friction, can be used to indicate transition and separation, and can be calibrated to cope with large variations in ambient or operating temperature.

The spectral density of skin friction in a turbulent boundary layer on a flat plate has been measured, although the calibration of the probe was restricted to frequencies below 100 c/s.

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Appendix 1

By considering steady two-dimensional laminar boundary-layer flow, it may be shown (Curle 1962, Howarth 1953) that the thermal-energy integral equation is

$$\frac{d}{dx}\int_0^\infty u(T-T_1)\,dy = -\frac{Q_w(x)}{\rho C_p},$$

where T_1 is the value of T as $y \to \infty$. Assume

$$T-T_1 = (T_w - T_1)\,(1+\lambda\eta + b\eta^2 + c\eta^3 + d\eta^4 \ldots),$$

where T_w is the value of T at $y = 0$ and $\eta \equiv y/\delta_T$, δ_T being the thermal-boundary-layer thickness and x-dependent. $\lambda, b, c, d, \ldots$ are functions of x only. The boundary conditions are

$$T = T_1, \quad \frac{\partial T}{\partial y} = 0, \ldots \quad \text{as} \quad \eta \to 1,$$

$$T = T_w, \quad \frac{\partial T}{\partial y} = \frac{Q_w}{k}, \quad \frac{\partial^2 T}{\partial y^2} = 0, \ldots \quad \text{as} \quad \eta \to 0.$$

If the thermal boundary layer is much thinner than the velocity boundary layer, which can be represented by terms up to η^4, then the above boundary conditions may be sufficient to determine the temperature profile completely, and the thermal-energy equation can be reserved for a relationship between heat transfer and skin friction. Applying the boundary conditions to

$$(T-T_1)/(T_w - T_1) = f(\eta)$$

gives

$$f(\eta) = (1 - 4\eta^3 + 3\eta^4) + \lambda(\eta - 3\eta^3 + 2\eta^4),$$

where λ is a shape parameter of the temperature profile

$$\lambda = \frac{Q_w \delta_T}{k(T_w - T_1)}.$$

Then $$\int_0^\infty y(T-T_1)\,dy = \int_0^1 \delta_T^2\eta(T_w-T_1)f(\eta)\,d\eta = ak^2(T_w-T_1)^3/Q_w^2,$$

where $$a = \lambda^2\int_0^1 \eta f(\eta)\,d\eta,$$

and $$\int_0^\infty y^2(T-T_1)\,dy = \int_0^1 \delta_T^3\eta^2(T_w-T_1)f(\eta)\,d\eta = -bk^3(T_w-T_1)^4/Q_w^3,$$

where $$b = -\lambda^3\int_0^1 \eta^2 f(\eta)\,d\eta.$$

Curle has shown that a and b may be taken as constant. Curle's values of a and b, using the similarity solutions, are 0·2226 and 0·1046, respectively.

Assume $$\mu u = \tau_w y + \tfrac{1}{2}(dp/dx)y^2.$$

Then $$\frac{d}{dx}\left\{\tau_w\int_0^\infty y(T-T_1)\,dy + \tfrac{1}{2}\frac{dp}{dx}\int_0^\infty y^2(T-T_1)\,dy\right\} = -\frac{\mu^2}{\rho k\sigma}\,Q_w(x),$$

and hence
$$\frac{d}{dx}\left\{\frac{ak^2\tau_w(T_w-T_1)^3}{Q_w^2} - \frac{b(dp/dx)k^3(T_w-T_1)^4}{2Q_w^3}\right\} = -\frac{\mu^2}{\rho k\sigma}Q_w(x). \qquad \text{(A 1.1)}$$

Assume that $Q_w(x)$, $T_w(x)$ are step functions, i.e.

for $0 < x < \xi$, $Q_w = 0$, $T_w - T_1 = 0$,
for $\xi \leqslant x \leqslant \xi + L$, $Q_w(x) = Q_w(\text{const.})$, $T_w - T_1 = \Delta T_0\,(\text{const.})$,
for $x > \xi + L$, $Q_w = 0$, $T_w - T_1 = 0$.

Then $$ak^2\tau_w\left(\frac{\Delta T_0}{Q_w}\right)^3 - \tfrac{1}{2}b\frac{dp}{dx}k^3\left(\frac{\Delta T_0}{Q_w}\right)^4 = -\frac{\mu^2 L}{\rho k\sigma}. \qquad \text{(A 1.2)}$$

Equation (A 1.2) is valid if (i) the thermal-boundary-layer thickness $\ll$ velocity-boundary-layer thickness, (ii) the substrate is sufficiently non-conducting to regard $Q_w(x)$, $T_w(x)$ as step functions.

Appendix 2. Probe calibration for a wide range of temperature difference

For the region near the wall where the velocity boundary layer may be considered to be linear, $\mu u = \tau_w y$.

If the thermal boundary layer produced by the film has thickness less than the linear part of the velocity layer, then

$$ak^2\tau_w(\Delta T_0/Q_w)^3 = -\mu^2 L/(\rho k\sigma), \qquad \text{(A 2.1)}$$

where a is a constant for a given temperature profile. (Curle's value is $a = 0{\cdot}2226$ for laminar velocity boundary layers, using the similarity solutions.) The parameter a may have different values in laminar and turbulent boundary layers.

Rearrangement of (A 2.1) gives

$$\tau_w^{\frac{1}{3}} = \left(\frac{-Q_w LW}{k\Delta T_0}\right)\frac{1}{a^{\frac{1}{3}}WL^{\frac{2}{3}}}\left(\frac{\mu\nu}{\sigma}\right)^{\frac{1}{3}}, \qquad \text{(A 2.2)}$$

where L, W are effective length and width of the film. In general μ, ν, k will vary across the thermal layer, but $k^{-1}(\mu\nu/\sigma)^{\frac{1}{3}}$ is constant for a wide range of temperatures, so the problem of choosing mean values is avoided. (The change in $k^{-1}(\mu\nu/\sigma)^{\frac{1}{3}}$ from 20 to 100 °C is 0·57 % approximately. Temperature differences are about 25 °C, incurring changes in $k^{-1}(\mu\nu/\sigma)^{\frac{1}{3}}$ across the thermal layer of approximately 0·18 %.)

The electrical heating of the film is i^2R. This heat is dissipated in two ways, $(-Q_w LW)$ is given to the airstream, and the remainder is given to the film substrate. L and W must be considered temperature-difference dependent, so that $(-Q_w LW) = Ci^2R + D$ where C, D may be functions of ΔT_0. Thus

$$\{-Q_w LW/(L^{\frac{2}{3}}W)\} = C_1 i^2R + D_1,$$

where C_1, D_1 may be functions of ΔT_0.

Taking $$C_1 = A_0 + A_1\Delta T_0, \quad D_1 = B_0 + B_1\Delta T_0,$$

$$\left(\frac{1}{L^{\frac{2}{3}}W}\right)\left(\frac{-Q_w LW}{\Delta T_0}\right) = \frac{A_0 + A_1\Delta T_0}{\Delta T_0} i^2R + \frac{B_0 + B_1\Delta T_0}{\Delta T_0}. \tag{A 2.3}$$

Substitution of (A 2.3) in (A 2.2) gives

$$\Delta T_0 \tau_w^{\frac{1}{3}} = k^{-1}(\mu\nu/a\sigma)^{\frac{1}{3}}\{(A_0 + A_1\Delta T_0)i^2R + B_0 + B_1\Delta T_0\}.$$

As $i \to 0$ the film is no longer hotter than the outside air and $\Delta T \to 0$. Thus $i^2R \propto \Delta T_0$ and hence $B_0 = 0$. Now put

$$\frac{1}{C_0} \equiv \frac{1}{k}\left(\frac{\mu\nu}{a\sigma}\right)^{\frac{1}{3}}$$

so that $$\frac{i^2R}{\Delta T_0} = \left(\frac{C_0}{A_0}\tau_w^{\frac{1}{3}} - \frac{B_1}{A_0}\right) + O(\Delta T_0). \tag{A 2.4}$$

This appears to be adequate for at least small variations in ΔT_0. Allowing for higher order terms,

$$i^2R/\Delta T_0 = \tau_w^{\frac{1}{3}}(A + B\Delta T_0) + C + D\Delta T_0, \tag{A 2.5}$$

where A, B, C, D are constants to be determined by calibration. If a linear resistance law is used to obtain ΔT_0, any errors incurred by ignoring the quadratic term will be absorbed in B and D. Thus ΔT_0 could be replaced by

$$\{(R/R_0) - 1\}\,10^3 \equiv z.$$

The method of calibration is:

(1) Keep τ_w constant.
(2) Alter ΔT_0 throughout its usable range.
(3) Plot i^2R against ΔT_0 or $\{(R/R_0) - 1\}\,10^3$.
(4) Repeat for other values of τ_w.

Appendix 3. Steady heat transfer from a hot element in the boundary layer

The thermal energy integral equation is

$$\frac{d}{dx}\int_0^\infty u(T - T_1)\,dy = \frac{-Q_w(x)}{\rho Cp}. \tag{A 3.1}$$

Take the temperature profile to be

$$T - T_1 = \Delta T \theta(\eta), \qquad \text{(A 3.2)}$$

where $\eta \equiv y/\delta_T$. Also,

$$Q_w(x) = k(\partial T/\partial y)_0 = O(k\Delta T_0/\delta_T) = \lambda k \Delta T_0/\delta_T. \qquad \text{(A 3.3)}$$

Consider a thin, small hot plate of length L, parallel to, and distance y from, the surface. For a sufficiently thin thermal layer,

$$u = u_0 + (\partial u/\partial z)_0 z + O(z^2). \qquad \text{(A 3.4)}$$

For the upper surface,

$$\frac{d}{dx}\int_0^\infty \left(u_0 + \left(\frac{\partial u}{\partial z}\right)_0 z\right) \Delta T_0 \theta(\eta)\, dy = -\frac{Q_w(x)}{\rho Cp},$$

and hence $$\frac{d}{dx}\int_0^1 \left(u_0 + \left(\frac{\partial u}{\partial z}\right)_0 \eta \frac{\lambda k \Delta T_0}{Q_w}\right) \theta(\eta) \frac{\lambda k \Delta T_0^2}{Q_w}\, d\eta = -\frac{Q_w(x)}{\rho Cp},$$

where λ is a constant. Therefore

$$\frac{d}{dx}\left(a u_0 + b\left(\frac{\partial u}{\partial z}\right)_0 \left(\frac{k\Delta T_0}{Q_w}\right)\right) \frac{k\Delta T_0^2}{Q_w} = -\frac{Q_w(x)}{\rho Cp}.$$

Integrating with respect to x, as in appendix 1,

$$a u_0 + b\left(\frac{\partial u}{\partial z}\right)_0 \left(\frac{k\Delta T_0}{Q_w}\right) = -\frac{Lk}{\rho Cp}\left(\frac{Q_w}{k\Delta T_0}\right)^2, \qquad \text{(A 3.5)}$$

with a similar expression for the lower surface.

If $$(\partial u/\partial z)_0 \delta_T \ll u_0 \qquad \text{(A 3.6)}$$

then $$\sqrt{u_0} \propto \frac{Q_w}{k\Delta T_0} = A\frac{i^2 R}{\Delta T_0} + B \quad (A \text{ and } B \text{ const.}), \qquad \text{(A 3.7)}$$

which is King's law. Near the surface, $u_0 = \tau_w \bar{y}/\mu$, $(\partial u/\partial z)_0 = \tau_w/\mu$, and (A 3.6) fails if $\delta_T \tau/\mu \sim \bar{y}\tau/\mu$, i.e. if $\delta_T \sim \bar{y}$.

Thus King's law fails if $\delta_T \geqslant \bar{y}$; but if $\delta_T \gg \bar{y}$ then $u_0 \ll (\partial u/\partial z)_0 \delta_T$ and the first term of (A 3.5) can be dropped. Thus

$$b\frac{\tau_w}{\mu} = -\frac{Lk}{\rho Cp}\left(\frac{Q_w}{k\Delta T_0}\right)^3,$$

and hence $$\tau_w^{\frac{1}{3}} \propto \frac{Q_w}{k\Delta T_0}. \qquad \text{(A 3.8)}$$

King's law (A 3.7) is accurate for uniform flow, equation (A 3.8) is accurate at the surface. In the inner part of the boundary layer equation (A 3.5) is valid.